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Mathematical description of unilateral constraints for discrete mechanical systems

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Abstract

The paper deals with mathematical description of unilateral constraints imposed on displacements and velocities. Discrete mechanical system are analyzed. Mathematical description of the problem concerning the interaction between material points and unilateral constraints is presented. Unilateral constraints can be defined as a model of a phenomenon occurring during the process of interaction between a material body and a displacement restriction. It was assumed that the reaction forces characterizing this interaction are elastic and the body deformation is small comparing with its displacements. Mathematical description of constraints is composed of geometrical conditions imposed on the body motion as well as a relationship between the reaction force and the body configuration. In this paper, geometrical characteristics of constraints are described using multi-valued mappings. The study focuses attention on the formulation of the problem with the use of the concept of unilateral constraints described by variational inequalities.

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1. Introduction

Unilateral constraints are models of phenomena resulting from the interaction of the material body with its motion restrictions. It is generally assumed, that the reaction force characterizing this interaction is elastic and the body deformations are small comparing to its displacements. In order to describe the geometric restrictions of the motion,

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the notion of multi-valued mappings as well as contingent derivatives is applied in this paper (see Aubin and Ekeland [3], Aubin and Frankowska [4]). On this basis, it is possible to determine a set of admissible positions, velocities and accelerations of the system at any time instant.

The influence of constraints on the body motion arises when the position vector belongs to the boundary of the set of admissible positions and when the velocity vector belongs to the boundary of the set of admissible velocities (active constraints). There is also another case of active constraints possible when the position of the system is located on the frontier of admissible positions, while the velocity vector does not belong to the set of admissible velocities. Such situation implicates impact phenomenon.

Nomenclature

$\text{Fr } \Omega, \text{Int } \Omega$	frontier (boundary) and interior of the set Ω
$D\Omega(t, X)$	contingent derivative of multi-valued mapping Ω
$T_C(X, \Omega)$	tangent cone to Ω at X
$N_C(X, \Omega)$	normal cone to Ω at X
$\text{Ap } \Omega$	apex of the convex set Ω
$d_H(A, B)$	Hausdorff distance between two closed sets $A, B \in R^N$
$\Psi_\Omega(X)$	indicator functional of the set Ω at X
$\partial(\cdot)$	subdifferential

2. Description of constraints

Mathematical description of constraints is composed of geometrical conditions imposed on body motion. Moreover, it contains a relationship between reaction force and location of the body. In our paper geometrical characteristics of constraints will be described using multi-valued mapping Ω , defined as follows: $\Omega: R_+^1 \rightarrow \text{cl conv } P(Q)$, where $\text{cl conv } P(Q)$ denotes a set of closed, convex and non-empty sub-sets of an open set $Q \subset R^N$.

Any function $\mathbf{Y} \in C([t_o, t_{end}), R^N)$ obeys geometrical condition defined by unilateral constraints, $\mathbf{Y}(t) \in \Omega(t) \quad \forall t \in [t_o, t_{end})$. For such time instants when $\mathbf{Y}(t) \in \text{Fr } \Omega(t)$ constraints are active. If $\mathbf{Y}(t) \in \text{Int } \Omega(t)$ constraints are non-active.

During the process of active unilateral constraints a discontinuous change in velocities and accelerations occurs. Thus, in order to describe the motion of analyzed system we assume an absolute continuous function $\mathbf{X} \in C_{ab}([t_o, t_{end}), Q)$, not being twice differentiable at finite number of time instants $t_k \in [t_o, t_{end})$ but possessing the left and the right hand derivatives of the 1st and 2nd order.

The velocity change at discontinuity time instants when $\dot{\mathbf{X}}^-(t_k) \neq \dot{\mathbf{X}}^+(t_k)$ can be established based on the impact problem solution. The active constraints process is associated with reaction force formation. In order to evaluate the reaction force and generalized acceleration, the appropriate algebraic problem should be formulated. If constraints are non-active, the reaction force equals zero.

Let us analyze kinematical implications of unilateral constraints described by the following inclusion

$$\mathbf{X}(t) \in \Omega(t), \quad t \in [0, t_{end}) \quad (1)$$

Constraints imposed on displacements by Eq. (1) implicate limitation of velocities if $\mathbf{X}(t) \in \text{Fr } \Omega(t)$. In order to describe velocity constraints the definition of the contingent derivative of multi-valued mapping can be applied (Grzesikiewicz et al. [12]). Taking into account the remarks made in [13], the formula for the contingent derivative of the mapping Ω is as follows

$$D\Omega(t_o, X_o) := \left\{ V \in R^N : \liminf_{h \rightarrow +0} \text{dist} \left(V, \frac{\Omega(t_o + h) - X_o}{h} \right) = 0 \right\} \subset R^N \quad (2)$$

The contingent derivative defined by Eq. (2) will be called differential successions of inclusion (1). It should be emphasized that if X_0 belongs to the interior of the set $\Omega(t_0)$ then $D\Omega(t_0, X_0) = R^N$ for $X_0 \in \text{Int } \Omega(t_0)$. Eq. (2) determines a set of vectors $V \in R^N$, implicating admissible right hand derivative of the function \mathbf{X} , i.e.

$$\dot{\mathbf{X}}^+(t) \in D\Omega(t, \mathbf{X}(t)), \quad t \in [0, t_{\text{end}}) \quad (3)$$

The above inclusion (3) constitutes the differential succession (kinematic implication) of the inclusion defined by Eq. (1). Inclusion (3) determines velocity constraints implicated by unilateral constraints defined by Eq. (1).

For any point $X \in R^N$, belonging to the set $\Omega(t)$, $t \in [0, t_{\text{end}})$, it is possible to associate a tangent cone. Let us remind that a cone in R^N is a set $\text{cone} \subset R^N$ if the relation $Y \in \text{cone}$ implicates $\lambda Y \in \text{cone}$ for any $\lambda > 0$.

The analysis of various definitions of tangent cones was carried out by Aubin and Ekeland [3]. It was proved in [3] that all these definitions are equivalent. In our paper we present a definition of tangent cone at point $X \in \Omega(t)$ having the form of a closed set:

$$T_C(X, \Omega(t)) := \text{cl} \left\{ W \in R^N : \liminf_{h \rightarrow +0} \text{dist} \left(W, \frac{\Omega(t) - X}{h} \right) = 0 \right\} \quad (4)$$

In case of the analyzed set $\Omega(t)$ the tangent cone described by Eq. (4) is closed and convex. Moreover, for any point belonging to the interior of the set $\Omega(t)$, the cone equals R^N . Thus, $T_C(X, \Omega(t)) = R^N$ if $X \in \text{Int } \Omega(t)$.

In analytical mechanics, any vector $V \in D\Omega(t_0, X_0)$ is called possible velocity at time instant $t_0 \in [0, t_{\text{end}})$ and for a position $X_0 \in \Omega(t_0)$ while any vector $W \in T_C(X, \Omega(t))$ is called virtual velocity (see Pars [26], Udwadia and Kalaba [33]).

Tangent cone to a convex and closed set can be also defined as a set of virtual displacements δx , as follows:

$$T_C(x, \Omega(t)) := \text{cl} \left\{ \delta x \in R^N : \delta x = \lambda (\xi - x), \quad \xi \in \Omega_t, \quad \lambda > 0 \right\} \quad (5)$$

For sets $D\Omega(t, X)$ and $T_C(X, \Omega(t))$ we assign the following set

$$\text{Ap } D\Omega(t, X) := \left\{ W \in R^N : \limsup_{\tau \rightarrow +0} d_H \left(\frac{\Omega(t+\tau) - X}{\tau}, W + T_C(X, \Omega(t)) \right) \geq 0 \right\} \quad (6)$$

where the function $d_H(A, B)$ denotes so called Hausdorff distance between two closed sets $A, B \in R^N$ (see Acary and Brogliato [1])

$$d_H(A, B) := \max \left\{ \sup_{X \in A} \text{dist}(X, B), \sup_{X \in B} \text{dist}(X, A) \right\} \quad (7)$$

Elements of the set $\text{Ap } D\Omega(t, X)$ will be called apexes of $D\Omega(t, X)$.

In an unpublished study A. Wakulicz proved that the set $D\Omega(t, X)$ can be presented as the following algebraic sum: $D\Omega(t, X) = \text{Ap } D\Omega(t, X) + T_C(X, \Omega(t))$ what leads to the following conclusions: (i) if $W \in \text{Ap } D\Omega(t, X)$ and $V \in W + T_C(X, \Omega(t))$ then $V \in D\Omega(t, X)$; (ii) if $W \in \text{Ap } D\Omega(t, X)$ and $V \in D\Omega(t, X)$ then $V \in W + T_C(X, \Omega(t))$.

In many cases the set defined by Eq. (6) contains only one element. Thus, $\text{Ap } D\Omega(t, X) = \{W\}$, $W \in R^N$ gives $D\Omega(t, X) = W + T_C(X, \Omega(t))$.

For many problems of dynamics only stationary constraints are considered, i.e. $\Omega(t) = \Omega_0$, $t \in [0, t_{\text{end}})$. In such a case the following relation holds $D\Omega_0(X) = T_C(X, \Omega_0)$, $X \in \Omega_0$.

The velocity constraints described by Eq. (3) implicate acceleration restrictions if $\ddot{\mathbf{X}}(t) \in \text{Fr } D\Omega(t, \mathbf{X}(t))$, $t \in [0, t_{\text{end}})$. In order to describe acceleration constraints the definition of the second differential succession of the inclusion (1) should be formulated

$$D^2 \Omega(t, X, V) := \left\{ a \in R^N : \liminf_{\substack{h \rightarrow 0+ \\ w \rightarrow V}} \text{dist} \left(a, \frac{D\Omega(t+h, X+hw) - V}{h} \right) = 0 \right\} \quad (8)$$

Equation (8) determines the set of admissible accelerations implicated by inclusion (1) if $X \in \Omega(t)$ and $V \in D\Omega(t, X)$. It should be emphasized that if $X \in \text{Int } \Omega(t)$ or $X \in \text{Fr } \Omega(t)$ and $V \in \text{Int } D\Omega(t, X)$ then $D^2 \Omega(t, X, V) = R^N$. Thus, the acceleration constraints are active if $X \in \text{Fr } \Omega(t)$ and $V \in \text{Fr } D\Omega(t, X)$. In such a case the set of admissible accelerations $D^2 \Omega(t, X, V)$, has the following form: $D^2 \Omega(t, X, V) = \text{Ap } D^2 \Omega(t, X, V) + T_C(V, D\Omega(t, X))$ what means that the set of virtual accelerations has a form of the cone $T_C(V, D\Omega(t, X))$. In case of stationary constraints, it gives $D^2 \Omega(X, V) = T_C(V, D\Omega(X))$.

3. Reaction of constraints

Taking into account both unilateral constraints described by Eq. (1) as well as kinematical implications defined via Eq. (2) and Eq. (8) leads to three different states of mechanical system. Any state of mechanical system at any time instant $t \in [0, t_{\text{end}})$ is determined by the position vector $\mathbf{X}(t) \in R^N$ and the velocity vector $\dot{\mathbf{X}}(t) \in R^N$. The first possible state of mechanical system occurs if $\mathbf{X}(t) \in \text{Int } \Omega(t)$ or if $\mathbf{X}(t) \in \text{Fr } \Omega(t) \wedge \dot{\mathbf{X}}(t) \in \text{Int } D\Omega(t, \mathbf{X}(t))$ then constraints are non-active.

In case of the second possible state of the system, if $\mathbf{X}(t) \in \text{Fr } \Omega(t) \wedge \dot{\mathbf{X}}(t) \in \text{Fr } D\Omega(t, \mathbf{X}(t))$ constraints are active what means that restrictions are imposed on acceleration vector. In such a case, the acceleration vector should obey the following relation $\ddot{\mathbf{X}}(t+0) \in D^2 \Omega(t, \mathbf{X}(t), \dot{\mathbf{X}}(t))$. As an effect of constraints, a reaction force appears. The reaction force should adjust accelerations to constraints.

The third possible state of the system is related to the following condition $\mathbf{X}(t) \in \text{Fr } \Omega(t) \wedge \dot{\mathbf{X}}(t-0) \notin D\Omega(t, \mathbf{X}(t))$. Then constraints are active and the impact phenomenon is observed. As a result of impact a discontinuous change in velocity occurs, such that $\dot{\mathbf{X}}(t+0) \in D\Omega(t, \mathbf{X}(t))$. The problem of impact will not be considered in our paper. It was partially analyzed in [13] but it requires more profound description in separate paper.

Adapting the body motion to constraints can be carried out by an additional force $r \in R^N$, which should be inserted to equations of motion. The reaction force $r \in R^N$ can be evaluated using various methods (see Grzesikiewicz [8]). In our paper only the perfect realization will be considered (see Arnold [2], Grzesikiewicz et al. [13]), what means that the reaction force is perfectly elastic. According to this assumption, it is possible to associate the following energetic indicator functional with constraints defined by Eq. (1) (see Panagiotopoulos [9])

$$\Psi_\Omega(X) := \begin{cases} 0 & \text{gdy } X \in \Omega \\ +\infty & \text{gdy } X \notin \Omega \end{cases} \quad (9)$$

The analyzed reaction force is described by the following inclusion

$$r \in -\partial \Psi_\Omega(X), \quad X \in \Omega(t) \quad (10a)$$

where $\partial \Psi_\Omega$ denotes subdifferential of the functional Ψ_Ω (Aubin and Ekeland [3]), defined as follows

$$\partial \Psi_\Omega(X) := \left\{ r \in R^N : \Psi_\Omega(\xi) - \Psi_\Omega(X) \geq r^T(\xi - X) \quad \forall \xi \in \Omega \right\} \quad (10b)$$

Taking into account the considerations presented by Aubin and Frankowska [4], one can prove (see Grzesikiewicz and Wakulicz [11]), that the relationships presented in Eq. (10a) and Eq. (10b) can be re-written to the following form

$$r \in N_C(X, \Omega(t)) \quad (11a)$$

$$N_C(X, \Omega(t)) := \{r \in R^N : r^T \xi \geq 0 \quad \forall \xi \in T_C(X, \Omega(t))\} \quad (11b)$$

The set of reactions $N_C(X, \Omega(t))$ described by Eq. (11b) has a form of a cone being orthogonal to the tangent cone $T_C(X, \Omega(t))$. The tangent cone $T_C(X, \Omega(t))$ constitutes the set of virtual velocities being called virtual displacements for static problems. As we assumed, the set Ω is convex and closed. Thus, Eq. (11b) describing admissible set of reactions can be presented in the following equivalent form

$$N_C(X, \Omega(t)) := \{r \in R^N : r^T(\eta - X) \geq 0 \quad \forall \eta \in \Omega(t)\} \quad (12)$$

The illustration of Eqs. (11) and (12) is given in Fig. 1. The set Ω determining unilateral constraints is visualized in Fig. 1a. For a certain point A at a time instant $t = t_0$, the tangent cone was established $T_C(X_A, \Omega(t_0))$ and depicted in Fig. 1b. Figure 1b contains also the normal cone $N_C(X_A, \Omega(t_0))$ representing the set of reactions and defined via Eq. (11b). Moreover, Fig. 1b visualizes the set $N_C^*(X_A, \Omega(t_0))$ being outer orthogonal to the tangent cone, i.e. $N_C^*(X_A, \Omega(t_0)) = -N_C(X_A, \Omega(t_0))$.

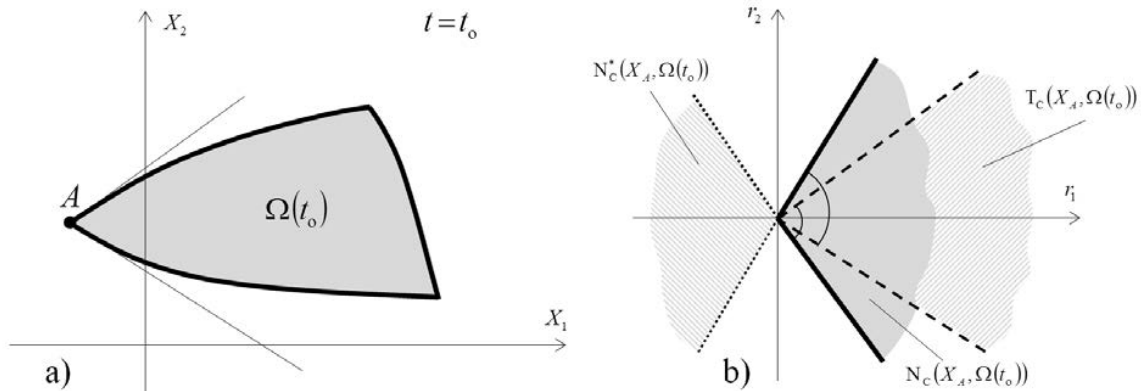


Fig. 1. Set of admissible positions (a). Tangent and normal cones at point A (b).

Let us note that the variational inequalities from Eq. (11b) and Eq. (12) describe the concept of virtual work being as follows:

$$r^T \xi \geq 0 \quad \forall \xi \in T_C(X, \Omega(t)) \quad (13a)$$

$$r^T(\eta - X) \geq 0 \quad \forall \eta \in \Omega(t) \quad (13b)$$

Consequently, if $X \in \text{Int } \Omega(t)$ then the variational inequality from Eq. (13b) gives $r = 0$. In theoretical mechanics Eqs. (13) determine reaction forces acting on mechanical system in equilibrium.

Description of reaction of constraints presented above relates reaction force $r \in R^N$ to admissible position $X \in \Omega(t) \subset R^N$. Such relationship in case of static problems allows to determine both the position vector and the reaction force.

The static problem with stationary constraints can be formulated as follows:

$$\begin{aligned} KX &= f + r \\ X &\in \Omega \\ r &\in N_C(X, \Omega) \end{aligned} \quad (14)$$

where K denotes stiffness matrix and f is a vector of generalized force.

Taking into account Eq. (10a) along with Eq. (12) or Eq. (13b) leads to a minimization formulation of the problem defined via Eqs. (14) as follows:

$$X = \arg \min \left\{ \frac{1}{2} \xi^T K \xi - f^T \xi + \Psi_{\Omega}(\xi) \right\} \quad (15)$$

where the minimized argument defines the energetic functional of mechanical system.

On the other hand, the above problems (14) and (15) can also be re-formulated to the following form

$$X = \arg \min_{\xi \in \Omega} \left\{ \frac{1}{2} \xi^T K \xi - f^T \xi \right\} \quad (16)$$

In case of Eq. (16) the argument of minimized function does not contain any non-differentiable term. It can be proved that the three static problems defined via Eqs. (14) and Eq. (15) and Eq. (16) are equivalent.

In case of dynamic problems related to mechanical systems with constraints, the main objective is to determine the acceleration vector of the system $\ddot{X} \in R^N$ as well as the reaction vector $r \in R^N$. In order to obtain the solution of such a problem, the equation of motion should be completed by a relationship between the vectors of reaction and acceleration.

4. Final remarks

Mathematical description of unilateral constraints for discrete mechanical systems using the notion of convex analysis and non-smooth mechanics was presented in this paper. The methods applied herein can be also implemented for selected problems related to mechanical systems containing bilateral constraints defined by non-differentiable surfaces. Moreover, description of motion for mechanical systems with non-holonomic constraints can also be performed applying presented methodology. Finally, it is possible to extend presented description for constraints defined by non-convex sets.

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